Constructing Tight Gabor Frames Using CAZAC Sequences

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Frames

A *finite frame* for $\mathbb{C}^N$ is a set $\mathcal{F} = \{\varphi_k\}_{k=1}^M$ such that there exists constants $0 < A \leq B < \infty$ where

$$A\|v\|_2^2 \leq \sum_{k=1}^{m} |\langle v, \varphi_k \rangle| \leq B\|v\|_2^2$$

for any $v \in \mathbb{C}^N$. $\mathcal{F}$ is called a *tight frame* if $A = B$ is possible.

Theorem

$\mathcal{F}$ is a frame for $\mathbb{C}^N$ if and only if $\mathcal{F}$ spans $\mathbb{C}^N$.

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The Frame Operator

Definition

Let $\mathcal{F} = \{v_i\}_{i=1}^{M}$ be a frame for $\mathbb{C}^N$, $x \in \mathbb{C}^N$, and $c \in \mathbb{C}^M$.

(a) The *analysis operator*, $T : \mathbb{C}^N \to \mathbb{C}^M$, is given by:

$$T(x) = \{\langle v_i, x\rangle\}_{i=1}^{M}.$$

(b) The *synthesis operator*, $T^* : \mathbb{C}^M \to \mathbb{C}^N$, is given by:

$$T^*(c) = \sum_{i=1}^{M} c[i] v_i.$$

(c) The *frame operator*, $S : \mathbb{C}^N \to \mathbb{C}^N$, is given by $S = T^*T$, i.e.,

$$S(x) = \sum_{i=1}^{M} \langle v_1, x\rangle v_i.$$

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Reconstruction via Frames

Theorem

Let $\mathcal{F} = \{v_i\}_{i=1}^{M}$ be a frame for $\mathbb{C}^N$ and $x \in \mathbb{C}^N$. Then,

$$x = \sum_{i=1}^{M} \langle x, S^{-1} v_i \rangle v_i.$$

Theorem

If $\mathcal{F}$ is a tight frame with bound A, then $S = A\, Id_N$. In partciular, $S^{-1} = A^{-1} Id_N$.

Reconstruction via Frames

Definition

Let $\mathcal{F} = \{v_i\}_{i=1}^M$ be a frame for $\mathbb{C}^N$. If for every $x \in \mathbb{C}^N$ $\mathcal{G} = \{u_i\}_{i=1}^M$ satisfies

$$x = \sum_{i=1}^{M} \langle x, u_i \rangle v_i,$$

then $\mathcal{G}$ is said to be a *dual frame* for $\mathcal{F}$. Since $\{S^{-1}v_i\}_{i=1}^M$ always satisfies this, $\{S^{-1}v_i\}_{i=1}^M$ is called the *canonical dual frame* of $\mathcal{F}$.

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Gabor Frames

Definition

(a) Let $\varphi \in \mathbb{C}^N$ and $\Lambda \subseteq (\mathbb{Z}/N\mathbb{Z}) \times \widehat{(\mathbb{Z}/N\mathbb{Z})}$. The *Gabor system*, (φ, Λ) is defined by

$$(\varphi, \Lambda) = \{e_n \tau_m \varphi : (m, n) \in \Lambda\}.$$

(b) If (φ, Λ) is a frame for $\mathbb{C}^N$ we call it a Gabor frame.

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Adjoint Subgroup

Definition

Let $\Lambda \subseteq (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})\widehat{}$ and $H_0 = \{\ell : (0, \ell) \in \Lambda\} \subseteq (\mathbb{Z}/N\mathbb{Z})\widehat{}$. The *adjoint subgroup* of Λ, $\Lambda^\circ \subseteq (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})\widehat{}$, is defined by

$$\Lambda^\circ = \{(m, n) : e_\ell \tau_k e_n \tau_m = e_n \tau_m e_\ell \tau_k, \forall (k, \ell) \in \Lambda\}$$

Time-Frequency Shifts as a Basis for Linear Operators

Theorem

The set of normalized time-frequency shifts, $\left\{\frac{1}{\sqrt{N}} e_\ell \tau_k : (k, \ell) \in (\mathbb{Z}/N\mathbb{Z}) \times \widehat{(\mathbb{Z}/N\mathbb{Z})}\right\}$ *forms an orthonormal basis for the* N^2*-dimensional Hilbert-Schmidt space of linear operators on* $\mathbb{C}^N$*.*

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Janssen's Representation

Theorem

Let Λ be a subgroup of $(\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})\widehat{}$ and $\varphi \in \mathbb{C}^N$. Then, the (φ, Λ) Gabor frame operator has the form

$$S = \frac{|\Lambda|}{|G|} \sum_{(m,n)\in\Lambda^\circ} \langle \varphi, e_n \tau_m \varphi \rangle e_n \tau_m.$$

Wexler-Raz Criterion

Theorem

Let Λ *be a subgroup of* $(\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})\widehat{}$. *For Gabor systems* (φ, Λ) *and* (Ψ, Λ), *we have*

$$x = \sum_{(k,\ell)\in\Lambda} \langle x, e_\ell \tau_k \Psi \rangle e_\ell \tau k \phi$$

if and only if for every $(m, n) \in \Lambda^\circ$,

$$\langle \varphi, e_\ell \tau_k \Psi \rangle = |G|/|\Lambda| \delta_{(m,n),(0,0)}.$$

Outline

Gabor Frames

CAZAC Sequences

CAZAC Generated Gabor Frames

Gram Matrix Method

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Definition

Let $\varphi \in \mathbb{C}^N$. φ is said to be a *constant amplitude zero autocorrelation (CAZAC) sequence* if

$$\forall k \in (\mathbb{Z}/N\mathbb{Z}), |\varphi_k| = 1 \quad \text{(CA)}$$

and

$$\forall m \in (\mathbb{Z}/N\mathbb{Z}), m \neq 0, \frac{1}{N}\sum_{k=0}^{N-1} \varphi_{k+m}\overline{\varphi_k} = 0. \quad \text{(ZAC)}$$

Examples

Quadratic Phase Sequences

Let $\varphi \in \mathbb{C}^N$ and suppose for each k, φ_k is of the form $\varphi_k = e^{-\pi i p(k)}$ where p is a quadratic polynomial. The following quadratic polynomials generate CAZAC sequences:

- Chu: $p(k) = k(k-1)$
- P4: $p(k) = k(k-N)$, N is odd
- Odd-length Wiener: $p(k) = sk^2$, $\gcd(s, N) = 1$, N is odd
- Even-length Wiener: $p(k) = sk^2/2$, $\gcd(s, 2N) = 1$, N is even

Examples

Let p be prime. The *Legendre symbol* is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 0, & \text{if } a \equiv 0 \mod p \\ 1, & \text{if } a \equiv n^2 \mod p \text{ for some } , n \neq 0 \\ -1, & \text{otherwise} \end{cases}$$

Examples

Björck Sequences

Let p be prime and $\varphi \in \mathbb{C}^p$ be of the form $\varphi_k = e^{i\theta(k)}$. Then φ will be CAZAC in the following cases:

- If $p \equiv 1 \mod 4$, then,

$$\theta(k) = \left(\frac{k}{p}\right) \arccos\left(\frac{1-p}{1+\sqrt{p}}\right)$$

- If $p \equiv 3 \mod 4$, then,

$$\begin{cases} \arccos\left(\frac{1-p}{1+p}\right), & \text{if } \left(\frac{k}{p}\right) = -1 \\ 0, & \text{otherwise} \end{cases}$$

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Properties

- $\varphi \in \mathbb{C}^N$ is CAZAC if and only if $\widehat{\varphi}$ is CAZAC.
- If $\varphi \in \mathbb{C}^N$ is CAZAC, then so is
  - If $|c| = 1$, $c\varphi[k]$ (Rotation)
  - $\tau_m\varphi[k] = \varphi[k-m]$ (Translation)
  - $e_n\varphi[k] = e^{2\pi ikn/N}\varphi[k]$ (Modulation)
  - If $\gcd(j, N) = 1$, $\pi_j\varphi[k] = \varphi[jk]$ (Decimation)
  - $\overline{\varphi}[k]$ (Conjugation)

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Question

Given a length N, how many CAZAC sequences of length N (whose first term is 1) are there?

(Partial) Answer

- If $N = p$ prime, there are at most $\binom{p-1}{2p-2}$ CAZAC sequences. (Haagerup)
- If N is composite and is *not* square-free, then there are infinitely many. (Björck-Saffari)
- It is unknown whether there are finite or infinitely many if N is composite and square-free.

Connection to Hadamard Matrices

Definition

Let H be a complex-valued $N \times N$ matrix.

(a) H is called a *Hadamard matrix* if $H^*H = NId_N$.

(b) H is called a *circulant matrix* if for each $j \geq 2$, the j-th row is a translation of the first row by $j - 1$.

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Connection to Hadamard Matrices

Theorem

Let $\varphi \in \mathbb{C}^N$ and let H be the circulant matrix given by

$$H = \begin{bmatrix} \text{———} \ \varphi \ \text{———} \\ \text{———} \ \tau_1\varphi \ \text{———} \\ \text{———} \ \tau_2\varphi \ \text{———} \\ \cdots \\ \text{———} \ \tau_{N-1}\varphi \ \text{———} \end{bmatrix}$$

Then, φ is a CAZAC sequence if and only if H is Hadamard. In particular there is a one-to-one correspondence between CAZAC sequences and circulant Hadamard matrices.

Connection to Cyclic N-roots

Definition

$x \in \mathbb{C}^N$ is a cyclic N-root if it satisfies

$$\begin{cases} x_0 + x_1 + \cdots + x_{N-1} = 0 \\ x_0x_1 + x_1x_2 + \cdots + x_{N-1}x_0 = 0 \\ \cdots \\ x_0x_1x_2 \cdots x_{N-1} = 1 \end{cases}$$

Connection to Cyclic N-roots

Theorem

(a) *If $\varphi \in \mathbb{C}^N$ is a CAZAC sequence then,*

$$\left(\frac{\varphi_1}{\varphi_0}, \frac{\varphi_2}{\varphi_1}, \cdots, \frac{\varphi_0}{\varphi_{N-1}}\right)$$

is a cyclic N-root.

(b) *If $x \in \mathbb{C}^N$ is a cyclic N-root then,*

$$\varphi_0 = x_0, \varphi_k = \varphi_{k-1} x_k$$

is a CAZAC sequence.

(c) *There is a one-to-one correspondence between CAZAC sequences which start with 1 and cyclic N-roots.*

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DPAF and STFT

Definition

Let $\varphi, \psi \in \mathbb{C}^N$.

(a) The *discrete periodic ambiguity function* of φ, $A_p(\varphi)$, is defined by

$$A_p(\varphi)[m,n] = \frac{1}{N}\sum_{k=0}^{N-1} \varphi[k+m]\overline{\varphi[k]}e^{-2\pi ikn/N} = \frac{1}{N}\langle \tau_{-m}\varphi, e_n\varphi\rangle.$$

(b) The *short-time Fourier transform* of φ with window ψ, $V_\psi(\varphi)$, is defined by

$$V_\psi(\varphi)[m,n] = \langle \varphi, e_n\tau_m\psi\rangle.$$

Full Gabor Frames Are Always Tight

Theorem

Let $\varphi \in \mathbb{C}^N \setminus \{0\}$. and $\Lambda = (\mathbb{Z}/N\mathbb{Z}) \times \widehat{(\mathbb{Z}/N\mathbb{Z})}$. Then, (φ, Λ) is always a tight frame with frame bound $N\|\varphi\|_2^2$.

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Λ°-sparsity

Definition

Let $\varphi \in \mathbb{C}^N$, $\Lambda \subseteq (\mathbb{Z}/N\mathbb{Z}) \times \widehat{(\mathbb{Z}/N\mathbb{Z})}$, and Λ° be the adjoint subgroup of Λ. We say that $A_p(\varphi)$ is Λ°-*sparse* if for every $(m, n) \neq (0, 0) \in \Lambda^\circ$ we have $A_p(\varphi)[m, n] = 0$.

Λ°-sparsity and Tight Frames

Theorem

Let $\varphi \in \mathbb{C}^N \setminus \{0\}$ *and let* $\Lambda \subseteq (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})\widehat{}$ *be a subgroup.* (φ, Λ) *is a tight frame if and only if* $A_p(\varphi)$ *is* Λ°*-sparse. The frame bound is* $|\Lambda| A_p(\varphi)[0, 0]$.

Λ°-sparsity and Tight Frames

Proof

By Janssen's representation and using the definition of $A_p(\varphi)$ we have,

$$S = \frac{|\Lambda|}{N} \sum_{(k,\ell)\in\Lambda^\circ} \langle e_\ell \tau_k \varphi, \varphi \rangle e_\ell \tau_k = \frac{|\Lambda|}{N} \sum_{(k,\ell)\in\Lambda^\circ} \langle \tau_k \varphi, e_{-\ell} \varphi \rangle e_\ell \tau_k$$

$$= |\Lambda| \sum_{(k,\ell)\in\Lambda^\circ} A_p(\varphi)[-k,-\ell] e_\ell \tau_k = |\Lambda| \sum_{(k,\ell)\in\Lambda^\circ} A_p(\varphi)[k,\ell] e_{-\ell} \tau_{-k}.$$

If $A_p(\varphi)$ is Λ°-sparse, then S is $|\Lambda| A_p(\varphi)[0,0]$ times the identity.

Λ°-sparsity and Tight Frames

To prove that Λ°-sparsity is a necessary condition, we note that for S to be tight we need

$$S = |\Lambda| \sum_{(k,\ell)\in\Lambda^\circ} A_p(\varphi)[k,\ell] e_{-\ell}\tau_{-k} = A\, Id.$$

We can rewrite this condition into

$$\sum_{(k,\ell)\in\Lambda^\circ\setminus\{(0,0\}} |\Lambda| A_p(\varphi)[k,\ell] e_{-\ell}\tau_{-k} + (|\Lambda| A_p(\varphi)[0,0] - A) Id = 0.$$

Since time-frequency shifts are linearly independent, we must have that $A_p(\varphi)$ is Λ°-sparse and the frame bound is $|\Lambda| A_p(\varphi)[0,0]$. ■

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DPAF of Chu Sequence

$$A_p(\varphi_{\text{Chu}})[m,n] :$$

$$\begin{cases} e^{\pi i(m^2-m)/N}, & m \equiv n \bmod N \\ 0, & \text{otherwise} \end{cases}$$

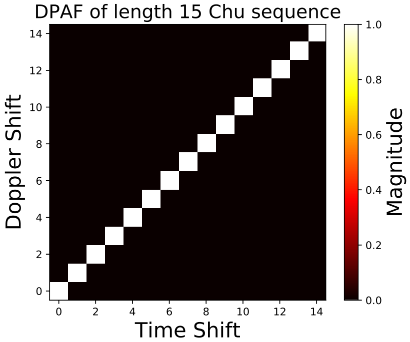

Figure: DPAF of length 15 Chu sequence.

Example: Chu/P4 Seqeunce

Proposition

Let $N = abN'$ *where* $\gcd(a, b) = 1$ *and* $\varphi \in \mathbb{C}^N$ *be the Chu or P4 sequence. Define* $K = \langle a \rangle$, $L = \langle b \rangle$ *and* $\Lambda = K \times L$.

(a) $\Lambda^\circ = \langle N'a \rangle \times \langle N'b \rangle$.

(b) (φ, Λ) *is a tight Gabor frame bound* NN'.

DPAF of Even Length Wiener Sequence

$$A_p(\varphi_{\text{Wiener}})[m,n] : \begin{cases} e^{\pi i s m^2/N}, & sm \equiv n \bmod N \\ 0, & \text{otherwise} \end{cases}$$

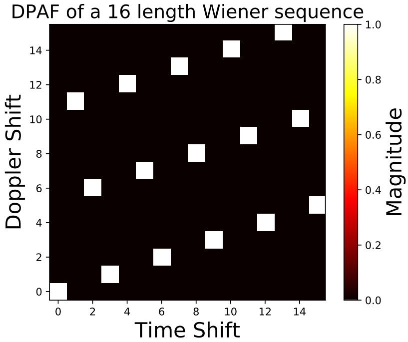

Figure: DPAF of length 16 P4 sequence.

DPAF of Björck Sequence

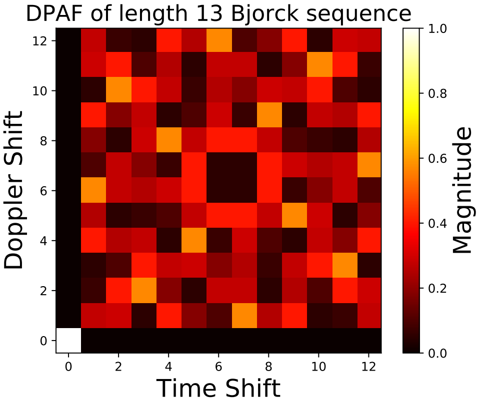

Figure: DPAF of length 13 Björck sequence.

DPAF of a Kronecker Product Sequence

Kronecker Product:
Let $u \in \mathbb{C}^M, v \in \mathbb{C}^N$.
$(u \otimes v)[aM + b] = u[a]v[b]$

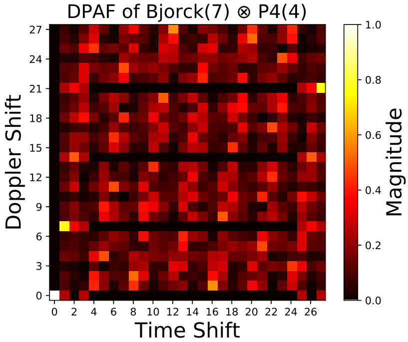

Figure: DPAF of Kroneker product of length 7 Bjorck and length 4 P4.

Example: Kronecker Product Sequence

Proposition

Let $u \in \mathbb{C}^M$ *be CAZAC,* $v \in \mathbb{C}^N$ *be CA, and* $\varphi \in \mathbb{C}^{MN}$ *be defined by the Kronecker product:* $\varphi = u \otimes v$*. If* $\gcd(M, N) = 1$ *and* $\Lambda = \langle M \rangle \times \langle N \rangle$*, then* (φ, Λ) *is a tight frame with frame bound* MN*.*

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Gram Matrix

Definition

Let $\mathcal{F} = \{v_i\}_{i=1}^{M}$ be a frame for $\mathbb{C}^N$. The *Gram matrix*, G, is defined by

$$G_{i,j} = \langle v_i, v_j \rangle.$$

This is the same as the linear operator given by TT^*, where T is the analysis operator.

Gram Matrix and DPAF

In the case of Gabor frames $\mathcal{F} = \{e_{\ell_m}\tau_{k_m}\varphi : m \in 0, \cdots, M-1\}$, we can write the Gram matrix in terms of the discrete periodic ambiguity function of φ:

$$G_{m,n} = Ne^{-2\pi i k_n(\ell_n - \ell_m)/N} A_p(\varphi)[k_n - k_m, \ell_n - \ell_m]$$

Rank of the Gram Matrix

Lemma

Let T be an $m \times n$ complex-valued matrix and let $G := TT^*$. Then, $rank(G) = rank(F)$.