A Delsarte-Style Proof of the Bukh–Cox Bound

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July 12, 2019

Outline

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Introduction

- The Bukh–Cox bound addresses the line packing problem: Pack n points in $\mathbb{R}\mathbf{P}^{d-1}$ ($\mathbb{C}\mathbf{P}^{d-1}$) that maximizes the minimum distance.
- Instances of this problem can be traced back to Tammes and F. Tóth.
- Most recent results identify new packings which achieve equality in the Welch Bound.
- Bukh and Cox discovered a different bound, along with a family of packings which achieve said bound.

Coherence

Definition

Let $X = \{x_i\}_{i \in [n]}$ be a sequence of unit vectors in $\mathbb{C}^d$. We define the *coherence* of X to be

$$\mu(X) := \max_{i \leq i < j \leq n} |\langle x_i, x_j \rangle|.$$

Essentially, $\mu(X)$ computes the smallest angle between any two vectors in X.

Welch Bound

Theorem (Welch '74)

Let $n > d$ and let $X = \{x_i\}_{i \in [n]}$ be a sequence of unit vectors in $\mathbb{C}^d$. Then,

$$\mu(X) \geq \sqrt{\frac{n-d}{d(n-1)}},$$

where equality is achieved if and only if X is an equiangular tight frame in $\mathbb{C}^d$.

Bukh–Cox Bound

Theorem (Bukh, Cox '18)

Let $n > d$ *and let* $X = \{x_i\}_{i \in [n]}$ *be a sequence of unit vectors in* $\mathbb{C}^d$*. Then,*

$$\mu(X) \geq \frac{(n-d)^2}{n + (n^2 - nd - n)\sqrt{1 + n - d} - (n-d)^2}.$$

Comparing the bounds

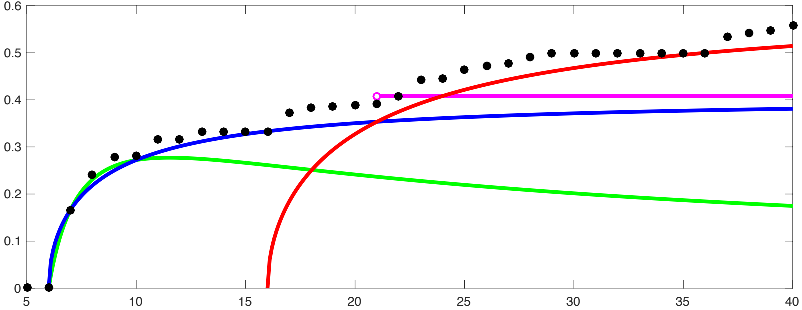

Figure 1: Coherence of best known packings in $\mathbb{R}^6$ for $5 \leq n \leq 40$ along with best known lower bounds. The Bukh–Cox bound is in green, and the Welch bound is in blue.

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Setup and Notation

- Let $X = \{x_i\}_{i \in [n]}$ be any sequence in $\mathbb{C}^d$, and identify X with the $d \times n$ matrix with columns x_i.

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- Define $\gamma(d,n) := \max_{Y\in T(d,n)} \|Y^*Y\|_1$.

Bukh–Cox Lemma

Lemma

Let $n = d + k$. Then, every $X \in N(d, n)$ satisfies

$$\mu(X) \geq \frac{n}{\|Y^*Y\|_1 - n} \geq \frac{n}{\gamma(k, n) - n}.$$

Furthermore, X minimizes $\mu(X)$ over $N(d, n)$ if X is equiangular and there exists $Y = \{y_i\}_{i \in [n]} \in T(k, n)$ such that

(i) *Y maximizes $\|Y^*Y\|_1$ over $T(k, n)$,*

(ii) *$XY^* = 0$, and*

(iii) *$sgn\langle x_i, x_j\rangle = -sgn\langle y_i, y_j\rangle$ for $1 \leq i < j \leq n$.*

Relation to Welch Bound

Theorem

For all $Y \in T(k,n)$ *we have*

$$\|Y^*Y\|_1 \leq n + \left[n(n-1)\left(\frac{n^2}{k} - n\right)\right]^{1/2}.$$

Equality is achieved if and only if Y *is an equiangular tight frame.*

Corollary

Let $n > d$. *For all* $X \in N(d,n)$,

$$\mu(X) \geq \sqrt{\frac{n-d}{d(n-1)}}.$$

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$\|Y^*Y\|_1$ bound for Bukh–Cox

Theorem

For all $Y \in T(k,n)$ *we have*

$$\|Y^*Y\|_1 \leq \frac{n^2(1+(k-1)\sqrt{1+k})}{k^2}.$$

Equality is achieved when Y *is of the form* $Y = [Z|Z|\cdots|Z]$, *where* $Z \in \mathbb{C}^{k\times k^2}$ *is an equiangular tight frame.*

Corollary

Let $n > d$. *For all* $X \in N(d,n)$,

$$\mu(X) \geq \frac{(n-d)^2}{n+(n^2-nd-n)\sqrt{1+n-d}-(n-d)^2}.$$

Special Polynomials

To prove the theorem, we will need the following special polynomials:

$$Q_0(x) = 1,$$
$$Q_1(x) = x - \frac{1}{k},$$
$$Q_2(x) = x^2 - \frac{4}{k+2}x + \frac{2}{(k+1)(k+2)}.$$

Proof of Theorem (1/6)

- Without loss of generality assume $y_i \neq 0$ for every $i \in [n]$. First, we normalize the columns of Y, $\{y_i\}_{i\in[n]}$, by defining $z_i := y_i/\|y_i\|_2$.
- The desired bound comes from finding a feasible set of coefficients for the following linear program:

$$\begin{aligned} \text{minimize} \quad & c_0 \\ \text{subject to} \quad & f(x) = c_0 Q_0(x) + c_1 Q_1(x) + c_2 Q_2(x), \\ & 0 \leq c_1 \leq k c_0, c_2 \leq 0, \\ & f(x) \geq \sqrt{x}, \forall x \in [0,1]. \end{aligned}$$

Proof of Theorem (2/6)

Suppose we have a feasible (c_0, c_1, c_2). Then,

$$\begin{aligned}
\|Y^*Y\|_1 &= \sum_{i=1}^{n}\sum_{j=1}^{n} |\langle z_i, z_j\rangle| \|y_i\|_2 \|y_j\|_2 \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n} f(|\langle z_i, z_j\rangle|^2) \|y_i\|_2 \|y_j\|_2 \\
&= \sum_{\ell=0}^{2} c_\ell \sum_{i=1}^{n}\sum_{j=1}^{n} Q_\ell(|\langle z_i, z_j\rangle|^2) \|y_i\|_2 \|y_j\|_2
\end{aligned}$$

Proof of Theorem (3/6)

Now we establish a bound for each innermost term $\ell = \{0, 1, 2\}$. Starting with $\ell = 0$, since $Q_0(x) = 1$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} Q_0(|\langle z_i, z_j\rangle|^2)\|y_i\|_2\|y_j\|_2 = \left(\sum_{i=1}^{n}\|y_i\|_2\right)^2 := S.$$

For $\ell = 1$, using $Q_1(x) = x - \frac{1}{k}$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} Q_1(|\langle z_i, z_j\rangle|^2)\|y_i\|_2\|y_j\|_2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|\langle y_i, y_j\rangle|^2}{\|y_i\|_2\|y_j\|_2} - \frac{S}{k}.$$

Proof of Theorem (4/6)

To bound the first term in the $\ell = 1$ case,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|\langle y_i, y_j\rangle|^2}{\|y_i\|_2\|y_j\|_2} \le \left(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|\langle y_i, y_j\rangle|^2}{\|y_i\|_2^2}\right)^{1/2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|\langle y_i, y_j\rangle|^2}{\|y_j\|_2^2}\right)^{1/2}$$
$$= \frac{n^2}{k}.$$

Overall, for $\ell = 1$ we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} Q_1(|\langle z_i, z_j\rangle|^2)\|y_i\|_2\|y_j\|_2 \le \frac{1}{k}(n^2 - S).$$

Proof of Theorem (5/6)

For the $\ell = 2$ case, let $\{e_m\}_{m=1}^{d_2}$ be an orthonormal basis for the (finite) vector space spanned by degree-4 projective harmonic polynomials in k varibles. There exists a constant $C_{d_2,k}$ such that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} Q_2(|\langle z_i, z_j\rangle|^2)\|y_i\|_2\|y_j\|_2 = C_{d_2k}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{d_2} e_m(z_i)\overline{e_m(z_j)}\|y_i\|_2\|y_j\|_2$$

$$= C_{d_2,k}\sum_{m=1}^{d_2}\left|\sum_{i=1}^{n} e_m(z_i)\|y_1\|_2\right|^2 \geq 0.$$

Multiplying both sides by $c_2 \leq 0$ gives

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_2 Q_2(|\langle z_i, z_j\rangle|^2)\|y_i\|_2\|y_j\|_2 \leq 0.$$

Proof of Theorem (6/6)

Putting all the bounds together gives

$$\begin{aligned}\|Y^*Y\|_1 &= \sum_{\ell=0}^{2} c_\ell \sum_{i=1}^{n}\sum_{j=1}^{n} Q_\ell(|\langle z_i, z_j\rangle|^2)\|y_i\|_2\|y_j\|_2 \\ &\le c_0 S + c_1 \frac{1}{k}(n^2 - S) \le c_0 n^2.\end{aligned}$$

Equality is achived if

1. $|\langle y_i, z_j\rangle| = |\langle z_i, y_j\rangle|, \forall i, j,$
2. $f(|\langle z_i, z_j\rangle|^2) = |\langle z_i, z_j\rangle|, \forall i, j,$
3. $\sum_i \sum_j Q_2(|\langle z_i, z_j\rangle|^2)\|y_i\|_2\|y_j\|_2 = 0,$
4. $\|y_i\|_2 = 1, \forall i,$

which occurs when Y is multiple copies of an ETF of k^2 vectors in $\mathbb{C}^k$.