Biangular Gabor Frames and Zauner's Conjecture

Mark Magsino
(Joint work with Dustin G. Mixon)

The Ohio State University

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Outline

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Frames

Definition

$F = \{f_j\}_{j=1}^n$ in $\mathbb{C}^d$ is a **frame** if

$$A\|x\|_2^2 \leq \sum_{j=1}^{n} |\langle x, f_j \rangle|^2 \leq B\|x\|_2^2 \quad \forall x \in \mathbb{C}^d$$

Furthermore, we say F is

- **tight** if $A = B$ is possible
- **unit norm** if $\|f_j\|_2 = 1$ for every j
- **equiangular** if $\exists \alpha \geq 0$ such that $|\langle f_j, f_{j'} \rangle|^2 = \alpha$ whenever $j \neq j'$

Zauner's Conjecture

Equiangular tight frames (ETFs) span optimally packed lines

Many applications:

- Compressed sensing
- Digital fingerprinting
- Multiple description coding

Important question: When do they exist?

Theorem (Gerzon bound)

There exists an ETF of n vectors in $\mathbb{C}^d$ only if $n \leq d^2$.

Zauner's Conjecture: For every d, there exists an ETF of d^2 vectors.

Gabor Frames

Definition

- **Translation** operator: $(Tv)(j) = v(j-1)$
- **Modulation** operator: $(Mv)(j) = e^{2\pi ij/d} \cdot v(j)$
- **Gabor** frame: $G(v) := \{M^\ell T^k v\}_{k,\ell=0}^{d-1}$
- **Fiducial** vector: v such that $G(v)$ is equiangular

Gabor frames are classically used in time–frequency analysis

Zauner's Conjecture (again): There's a fiducial vector in every $\mathbb{C}^d$ (!)

Constructive solutions to Zauner

Idea #1:

- Fiducial vectors = sol'ns to polynomial system
- Compute a Gröbner basis! $\Omega(2^{2^d})$ runtime...

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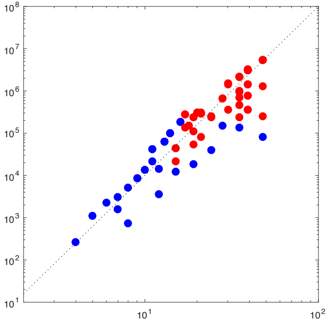
- Known fiducials have nice field structure (!)
- Round numerical sol'ns! $\Omega(d^4)$ description...



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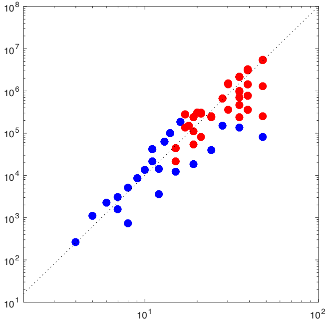
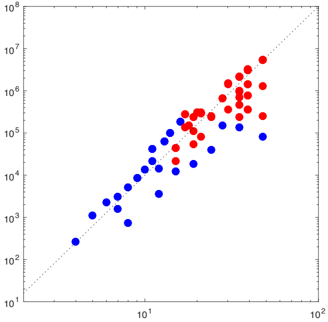


Idea #3:

- Stark units provide a shorter description of several known fiducials (!)
- Kopp (2018) leveraged this feature to find first known fiducial for $d = 23$
- Constructive proof of Zauner will likely be conditioned on Stark conjectures...

Constructive solutions to Zauner



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What about an unconditional proof? We'll need to be non-constructive...

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Biangular Gabor frames

Big idea: Relax to biangular frames and use intermediate value theorem

Definition

$G(v)$ is (α, β)-**biangular** if

(i) $|\langle v, T^k v\rangle|^2 = \alpha$ for $k \in \{1, \cdots, d-1\}$, and

(ii) $|\langle v, M^\ell T^k v\rangle|^2 = \beta$ for $k \in \{0, \cdots, d-1\}$ and $\ell \in \{1, \cdots, d-1\}$.

Lemma

If $G(v)$ is an (α, β)-biangular Gabor frame for $\mathbb{C}^d$, then $\alpha + d\beta = \|v\|_2^4$.

Examples

- $G(\mathbf{1})$ is $(d^2, 0)$-biangular

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- $G(\hat{f})$ is $(0, 1/d)$-biangular if f is the Alltop sequence

$$f(t) := \frac{1}{\sqrt{d}} e^{2\pi i t^3/d}$$

(requires prime $d \geq 5$, **"Gabor MUB"**)

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- $G(v)$ is biangular $\implies$ $G(cv)$ is biangular for $c \in \mathbb{C}^\times$

Real Algebraic Varieties

Define $B_d := \left\{ v \in \mathbb{C}^d \text{ for which } G(v) \text{ is biangular} \right\}$

Paradoxical observations:

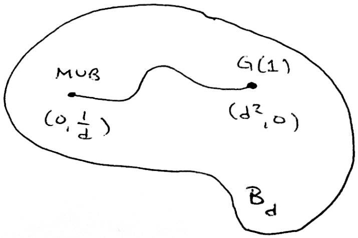
- B_d is defined by $\Omega(d^2)$ polynomials over $2d + 2$ real variables
- (Computer) For some d, $B_d/\mathbb{C}^\times$ is one-dimensional

Lemma

Given $d \in \mathbb{N}$, suppose

- *B_d is path-connected, and*
- *there exists a Gabor MUB in $\mathbb{C}^d$.*

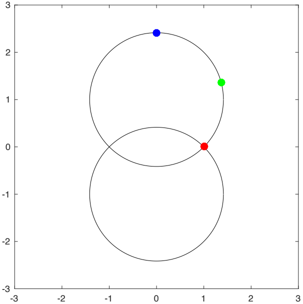
Then there exists a fiducial in $\mathbb{C}^d$.



Proof of concept: $d = 2$

It's convenient to define $C_d := \{v \in B_d : v(0) = 1\}$

Easy calculation: C_2 = union of two circles:



$$v_0 = \begin{bmatrix} 1 \\ (1+\sqrt{2})i \end{bmatrix}$$

$v^\star$ = fiducial

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Path-Connectedness

Open question: For which d is B_d path-connected?

Sufficient condition:

Lemma

C_d is path-connected $\implies$ B_d is path-connected

Related work:

- Cahill, Mixon, Strawn (2017): FUNTFs are connected
- Needham, Shonkwiler (2018): Symplectic geometry techniques

Remainder of this talk: Numerical evidence of path-connectivity

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Method

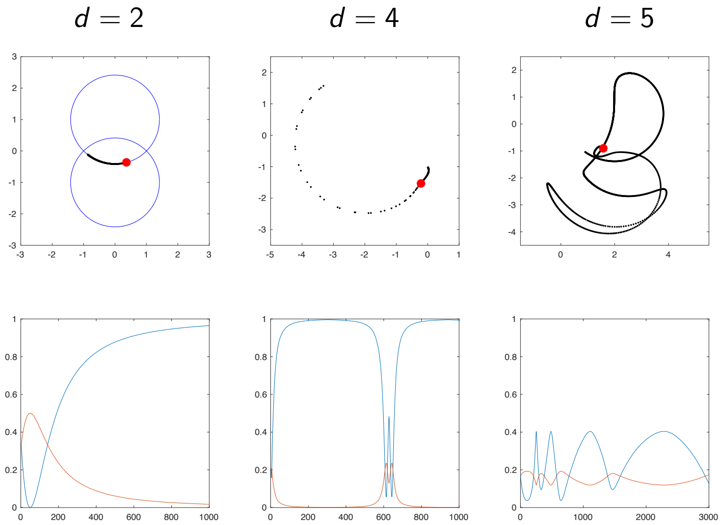
For each $d \in \{2, 4, 5\}$, do:

- Put $v_0 :=$ known numerical fiducial (Scott, Grassl 2010)
- Perturb v_0 and locally minimize $\sum \left(\text{defining polys}\right)^2$ to get v_1
- For each $j > 1$, locally minimize from the perturbation

$$v_j + c \cdot \frac{v_j - v_{j-1}}{\|v_j - v_{j-1}\|_2}$$

to get v_{j+1}

Results



$d = 2$
$d = 4$
$d = 5$
3
2
1
0
-1
-2
-3
-3 -2 -1 0 1 2 3
2
1
0
-1
-2
-3
-5 -4 -3 -2 -1 0 1
2
1
0
-1
-2
-3
-4
0 2 4
1
0.8
0.6
0.4
0.2
0
0 200 400 600 800 1000
0 200 400 600 800 1000
0 1000 2000 3000

Open Problems

- When is B_d path-connected?
- We don't really need Gabor MUBs! Instead:

  Find $v \in \mathbb{C}^d$ such that $G(v)$ is (α, β)-biangular with $\alpha < \frac{1}{d+1}$
- Can we use B_d to find more numerical fiducials?

Questions?

Thanks for listening!